

# GEOMETRIC PROPERTIES OF GELFAND'S PROBLEMS WITH PARABOLIC APPROACH

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**ABSTRACT.** We consider the asymptotic profiles of the nonlinear parabolic flows

$$(e^u)_t = \Delta u + \lambda e^u$$

to show the geometric properties of the following elliptic nonlinear eigenvalue problems known as a Gelfand's problem:

$$\begin{aligned} \Delta \varphi + \lambda e^\varphi &= 0, \quad \varphi > 0 \quad \text{in } \Omega \\ \varphi &= 0 \quad \text{on } \Omega \end{aligned}$$

posed in a strictly convex domain  $\Omega \subset \mathbb{R}^n$ . In this work, we show that there is a strictly increasing function  $f(s)$  such that  $f^{-1}(\varphi(x))$  is convex for  $0 < \lambda \leq \lambda^*$ , i.e., we prove that level set of  $\varphi$  is convex. Moreover, we also present the boundary condition of  $\varphi$  which guarantee the  $f$ -convexity of solution  $\varphi$ .

## 1. INTRODUCTION

We will investigate the geometric properties of parabolic flows and derive related geometric properties for the asymptotic limits of such evolutions. More precisely, we consider the nonnegative solutions  $u(x, t)$  of the following equation

$$(1.1) \quad (e^u)_t = \Delta u + \lambda e^u$$

posed on a strictly convex and bounded domain  $\Omega$  with zero boundary condition

$$(1.2) \quad u = 0 \quad \text{on } \partial\Omega,$$

and initial data

$$(1.3) \quad u(x, 0) = u_0(x) > 0$$

In the limit, these flows converge to solutions of the well-known Gelfand's problems

$$(GP_\lambda) \quad \begin{cases} \Delta \varphi(x) = -\lambda e^{\varphi(x)} & \text{in } \Omega \\ \varphi(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

By this relation, it is natural to expect that the solution of the parabolic flow above have a lot in common with those of the Gelfand's problems,  $(GP_\lambda)$ . The aim of the paper is to provide geometric properties of solutions of the Gelfand's problems by using the parabolic method which is introduced by Lee and Vázquez, [LV].

Recently, there are a lot of studies of the problem  $(GP_\lambda)$  because of its wide applications. It arises in many physical models: it describes problems of thermal self-ignition [Ge], a ball of isothermal gas in gravitational equilibrium proposed by Lord Kelvin [Ch], the problem of temperature distribution in an object heated by the application of a uniform electric current [KC] and Osanger's vortex model for turbulent Euler flows [CLMP].

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1991 *Mathematics Subject Classification.* Primary 35K55, 35K65.

*Key words and phrases.* Porous medium equation, large time behavior, eventual concavity, convergence of supports, Gelfand.

Let us summarize some known results to  $(GP_\lambda)$ . It is well-known that there exists a finite positive number  $\lambda^*$ , called the *extremal value*, such that Gelfand's problem  $(GP_\lambda)$  has at least a classical solution which is minimal among all possible positive solutions if  $0 \leq \lambda < \lambda^*$ , while no solution exists, even in the weak sense, for  $\lambda > \lambda^*$ . Let us call the minimal solution  $\underline{\varphi}_\lambda$ . The family of such solutions depends smoothly and monotonically on  $\lambda$ , and in particular

$$\underline{\varphi}_\lambda < \underline{\varphi}_{\lambda'} \quad \text{if } \lambda < \lambda'.$$

Their limit as  $\lambda \nearrow \lambda^*$  is the external solution  $\varphi_{\lambda^*}$  and it can be either classical or singular (i.e. unbounded). It is known that  $\varphi^* \in L^\infty(\Omega)$  for every  $\Omega$  if  $N \leq 9$ , while  $\varphi^* = \log\left(\frac{1}{|x|^2}\right)$  and  $\lambda^* = 2(n-2)$  if  $N \geq 10$  and  $\Omega = B_1$ . Brezis and Vázquez [BV] investigated the existence and regularities of extremal solutions when they are unbounded. The regularity theory for the Gelfand's problem was improved by S. Nedev [Ne] who proved that, for general smooth domain  $\Omega$ ,  $\varphi^*$  is a classical solution if  $n \leq 3$ , while  $\varphi^* \in H_0^1(\Omega)$  if  $n \leq 5$ .

The ultimate goal in this article is to establish the geometric properties of  $\underline{\varphi}_\lambda$  for  $\lambda < \lambda^*$ . Especially, we'd like to show that

$$f(\underline{\varphi}_\lambda) = e^{-\frac{1}{2}\underline{\varphi}_\lambda} : \quad \text{convex}$$

From now on, we call it  $f$ -convexity shortly. We also refer to the minimal solution  $\underline{\varphi}_\lambda$  of the Gelfand's problem  $(GP_\lambda)$  as  $\varphi$  for the rest of the paper. Since the minimal solution  $\underline{\varphi}_\lambda$  can be obtained as the limit of solution  $u$  to (1.1)-(1.3) as  $t \rightarrow \infty$ , we will concentrate on showing  $f$ -convexity of  $u$  under the assumption that the initial value  $u_0$  has the following property

$$f(u_0) = e^{-\frac{1}{2}u_0} : \quad \text{strictly convex,}$$

The parabolic approximation method introduced in [LV] relies on the fact that the nonlinear elliptic problem  $(GP_\lambda)$  can be describe the asymptotic profile of a corresponding parabolic flow in a bounded domain and we use that possibility as follows: we select an initial data for the parabolic flow having the desired geometric property. Then, the corresponding solution  $u$  of (1.1) will converge eventually to the minimal solution  $\varphi$  as  $t \rightarrow \infty$ . If the evolution preserves the  $f$ -convexity property under investigation, the result for the problem  $(GP_\lambda)$  will be obtained in the limit  $t \rightarrow \infty$ .

To investigate the  $f$ -convexity of solutions to the corresponding parabolic flow, we will split it into two steps. The step 1 will be devoted to the study of the  $f$ -convexity of solution  $u$  on the boundary. On  $\partial\Omega$ , the second derivatives of  $f(u)$  can be written in the form

$$[f(u)]_{\alpha\alpha} = \frac{1}{2}e^{-\frac{1}{2}u} \left( \frac{1}{2}u_\alpha^2 - u_{\alpha\alpha} \right), \quad (D_{e_\alpha} u = u_\alpha).$$

Hence, the geometric properties of solutions on the boundary can be determined by the balances between quantities,  $u_\alpha$  and  $u_{\alpha\alpha}$ . However, the difference between them is very subtle in this problem. Therefore, there is little room for perturbing quantities. Thus, it is very difficult to investigate the geometric properties of solution on the boundary without some boundary condition. In this paper, we will focus on the solutions of (1.1) having the conditions not only (1.2), (1.3) but also

$$(1.4) \quad G(u, \lambda, \Omega) = \frac{1}{2}u_\nu^2 + \lambda + (n-1)u_\nu H(\partial\Omega) + Ku_\nu > 0 \quad \text{on } \partial\Omega$$

for sufficiently large  $K > 0$  where  $\nu$  is the outer normal vector to  $\partial\Omega$  and  $H(\partial\Omega)$  is the mean curvature of  $\Omega$  at the boundary. Here the constant  $K$  is related to the shape of boundary  $\partial\Omega$ .

As the second step, we extend the geometric result on the boundary to the interior of  $\Omega$ . Since the continuity of the second derivatives of  $f(u)$ , it is expected that if there are some problems or difficulties then they may occur at the region far away from the boundary. Hence, the equation that

describe the parabolic flow plays a very important mathematical role in the study of the geometric properties of solution in the interior.

**1.1. Outlines.** This paper is divided into three parts: In Part 1 (Section 2) we study the convexity of solution to the degenerate equation

$$(1.5) \quad w_t = \rho(w)a^{ij}(\nabla w)w_{ij} + b(w)|\nabla w|^p + c(w)|\nabla w|^2 + d(w)$$

on a strictly convex domain  $\Omega$ . It is a simple observation that the convexity of solution will be strongly effected by the coefficients. Hence, proper conditions need to be imposed to the coefficients of (1.5) for the result. The Part 2 (Section 3) is devoted to the proof for the  $f$ -convexity of the minimal solution to the Gelfand's problem with boundary condition (1.4). As will be mentioned later, the main equation of Gelfand's problem is a special form of (1.5). Thus, the proof will be focused on the  $f$ -convexity of solution on the boundary. In Part 3, we will discuss the boundary condition (1.4) of a solution

## 2. CONVEXITY FOR DEGENERATE EQUATION

In this section, we will study degenerated equations of the form

$$(2.1) \quad w_t = \rho(w)a^{ij}(\nabla w)w_{ij} + b(w)|\nabla w|^p + c(w)|\nabla w|^2 + d(w), \quad (p \geq 2)$$

on the bounded cylinder  $\Omega \times [0, \infty)$ , where  $\Omega$  is a bounded strictly convex domain in  $\mathbb{R}^n$  with smooth boundary. The subindices  $i, j \in \{1, \dots, n\}$  denote differentiation with respect to the space variables  $x_1, \dots, x_n$  and the summation convention is used. We assume that the coefficient matrix  $(a^{ij})$  is strictly positive and all coefficients  $a^{ij}, b, c$  and  $d$  belong to appropriate  $C^k$ , ( $k = 1, \dots$ ) space which will be defined later. The degeneracy of the equation is carried through the function  $\rho(w)$  which is assumed to be smooth on  $\Omega$ .

We assume further that the coefficients of (2.1) satisfies the following conditions:

$$\text{I.1} \quad b(s), c(s) \text{ and } d(s) \text{ are convex,}$$

$$\text{I.2} \quad \rho'' - \frac{(\rho')^2}{2\rho} = 0.$$

Denoting by  $L$  the operator

$$Lw = w_t - \left( \rho(w)a^{ij}(\nabla w)w_{ij} + b(w)|\nabla w|^p + c(w)|\nabla w|^2 + d(w) \right), \quad (p \geq 2),$$

we can now state the main result in this section:

**Lemma 2.1.** *Let  $\Omega$  be a strictly convex bounded domain in  $\mathbb{R}^n$  with smooth boundary and suppose that the coefficients  $a^{ij}, b, c$  and  $d$  of the operator  $L$  are smooth and satisfy the elliptic condition*

$$a^{ij}\xi_i\xi_j \geq c_0|\xi|^2 > 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

*for some positive constant  $c_0$ . In addition, assume that  $\rho$  is a smooth function on  $\Omega$  and strictly positive in its interior. Let  $w$  be a smooth solution of (2.1) satisfying the conditions conditions **I.1-2**. If  $w$  is convex on the parabolic boundary of  $\Omega \times [0, \infty)$ , i.e., if  $\inf_{\partial_t \Omega \times [0, \infty)} \inf_{|e_\alpha|=1} w_{\alpha\alpha}(x, t) > 0$  and  $\inf_{\Omega} \inf_{|e_\alpha|=1} w_{\alpha\alpha}(x, 0) > 0$ , then  $w$  is convex in the space variable for all  $t > 0$ , i.e.,  $\inf_{\Omega \times [0, \infty)} \inf_{|e_\alpha|=1} w_{\alpha\alpha}(x, t) \geq 0$ .*

*Proof.* By direct computation, (2.1) implies that the evolution of  $w_{\alpha\beta}$  is given by the equation (2.2)

$$\begin{aligned}
w_{,\alpha\beta t} = & \rho(w)a^{ij}(\nabla w)w_{\alpha\beta,ij} + \rho(w)a_k^{ij}(\nabla w)(w_{\alpha k}w_{\beta ij} + w_{\beta k}w_{\alpha ij}) \\
& + \rho'(w)a^{ij}(\nabla w)(w_{\alpha}w_{\beta ij} + w_{\beta}w_{\alpha ij}) + \rho'(w)w_{\alpha}a_k^{ij}w_{\beta k}D_{ij}w + \rho'(w)w_{\beta}a_k^{ij}w_{\alpha k}w_{ij} \\
& + \rho(w)a_k^{ij}(\nabla w)w_{k\alpha\beta}w_{ij} + \rho(w)a_{kl}^{ij}(\nabla w)w_{k\alpha}w_{l\beta}D_{ij}w \\
& + \rho'(w)w_{\alpha\beta}a^{ij}(\nabla w)D_{ij}w + \rho''(w)w_{\alpha}w_{\beta}a^{ij}(\nabla w)D_{ij}w + b'|\nabla w|^p w_{\alpha\beta} \\
& + [pb(w)|\nabla w|^{p-2} + 2c(w)](\nabla w_{\alpha} \cdot \nabla w_{\beta} + \nabla w \cdot \nabla w_{\alpha\beta}) \\
& + p(p-2)b(w)w_k w_l w_{k\alpha} w_{l\beta} |\nabla w|^{p-4} + [pb'(w)|\nabla w|^{p-2} + 2c'(w)](w_{\alpha} \nabla w \cdot \nabla w_{\beta} + w_{\beta} \nabla w \cdot \nabla w_{\alpha}) \\
& + b''(w)|\nabla w|^{p-2} w_{\alpha} w_{\beta} + c'(w)w_{\alpha\beta}|\nabla w|^2 + c''(w)w_{\alpha}w_{\beta}|\nabla w|^2 + d'(w)w_{\alpha\beta} + d''(w)w_{\alpha}w_{\beta}
\end{aligned}$$

To estimate the minimum of the second derivatives of  $w$  with respect to space variables, we take a look at the following quantity, for a positive function  $\psi(t)$ ,

$$\inf_{x \in \Omega, s \in [0, t]} \inf_{e_{\beta} \in \mathbb{R}^n, |e_{\beta}|=1} [w_{\beta\beta} + \epsilon\psi(t)] = w_{\bar{\alpha}\bar{\alpha}}(x_0, t_0) + \epsilon\psi(t_0)$$

We need to show that there are a function  $\psi(t)$  and  $\epsilon_0$  such that

$$w_{\bar{\alpha}\bar{\alpha}}(x_0, t_0) + \epsilon\psi(t_0) > 0 \quad \forall 0 < \epsilon < \epsilon_0.$$

To get a contradiction, suppose that

$$(2.3) \quad w_{\bar{\alpha}\bar{\alpha}}(x_0, t_0) + \epsilon\psi(t_0) = 0.$$

Observe that the minimum is taken at  $(x_0, t_0)$  with direction  $\bar{\alpha}$ . By the previous lemma, the minimum point of  $w_{\alpha\alpha}$  is little way off the boundary. Hence, we can put  $x_0$  without loss of generality. On the other hand, the parabolic equation  $w_{\alpha\alpha}$  contains third order derivatives which is difficult to control with the information of  $w_{\alpha\alpha}$ . Hence we are going to perturb the direction of the derivative to create extra terms, keeping the minimum point and minimum zero.

We now use the function

$$Z = w_{,\alpha\beta}\eta^{\alpha}\eta^{\beta} + \epsilon\psi(t)|\eta|^2$$

where the modifying functions  $\eta^{\beta}(x)$  are constructed as follows: at  $x = 0$  we assume that the  $\eta^{\beta}$  satisfy the system

$$(2.4) \quad \eta_{,i}^{\beta} = (c_{\gamma}\eta^{\gamma})\delta_{\beta i}, \quad \eta_{,ij}^{\beta} = (c_l\eta^l)c_{\gamma}\delta_{\gamma i}\delta_{\beta j},$$

where the subscripts are space derivatives. Putting also  $\eta^{\beta}(0) = \delta_{\bar{\alpha}\beta}$ , it follows that

$$\eta_{,i}^{\beta}(0) = c_{\gamma}\delta_{\bar{\alpha},\gamma}\delta_{\beta i}, \quad \eta_{,ij}^{\beta}(0) = c_{\bar{\alpha}}c_{\gamma}\delta_{\gamma i}\delta_{\beta j},$$

and

$$\eta^{\beta}(x) = \delta_{\bar{\alpha}\beta} + c_{\bar{\alpha}}x^{\beta} + \frac{1}{2}c_{\bar{\alpha}}c_{\gamma}x^{\gamma}x^{\beta}.$$

Hence, at  $x = 0$ ,

$$\begin{aligned}
Z_{,i} &= w_{,\alpha\beta i}\eta^{\alpha}\eta^{\beta} + 2w_{,\beta i}c_{\alpha}\eta^{\alpha}\eta^{\beta} + \epsilon\psi(t)(|\eta|^2)_i \\
Z_{,ij} &= w_{,\alpha\beta ij}\eta^{\alpha}\eta^{\beta} + 2c_{\alpha}w_{,\beta ij}\eta^{\alpha}\eta^{\beta} + 2c_{\beta}w_{,\alpha ij}\eta^{\alpha}\eta^{\beta} \\
&\quad + 2c_jc_{\alpha}w_{,\beta i}\eta^{\alpha}\eta^{\beta} + 2c_ic_{\alpha}w_{,\beta j}\eta^{\alpha}\eta^{\beta} + 2c_{\alpha}c_{\beta}w_{,ij}\eta^{\alpha}\eta^{\beta} + \epsilon\psi(t)(|\eta|^2)_{ij}.
\end{aligned}$$

$$\begin{aligned}
(2.5) \quad Z_{,t} - \epsilon \psi' |\eta|^2 = & \rho a^{ij} D_{ij} Z + \rho a_k^{ij} [w_{\alpha k} w_{\beta ij} \eta^\alpha \eta^\beta + w_{\beta k} w_{\alpha ij} \eta^\alpha \eta^\beta] \\
& + a^{ij} [\rho' w_\alpha - 2\rho c_\alpha] w_{\beta ij} \eta^\alpha \eta^\beta + a^{ij} [\rho' w_\beta - 2\rho c_\beta] w_{\alpha ij} \eta^\alpha \eta^\beta \\
& + a^{ij} [\rho'' w_\alpha w_\beta - 2\rho c_\alpha c_\beta] w_{ij} \eta^\alpha \eta^\beta - 2\rho a^{ij} c_j c_\alpha w_{, \beta i} \eta^\alpha \eta^\beta - 2\rho a^{ij} c_i c_\alpha w_{, \beta j} \eta^\alpha \eta^\beta \\
& - \epsilon \psi a^{ij} (|\eta|^2)_{ij} + \rho a_k^{ij} w_{k\alpha\beta} w_{ij} \eta^\alpha \eta^\beta + \rho a_{ij}^{kl} w_{k\alpha} w_{l\beta} w_{ij} \eta^\alpha \eta^\beta \\
& + \rho' w_{\alpha\beta} a_{ij} w_{ij} \eta^\alpha \eta^\beta + \rho' a_k^{ij} w_\alpha w_{k\beta} w_{ij} \eta^\alpha \eta^\beta + \rho' a_{ij}^k w_\beta w_{k\alpha} w_{ij} \eta^\alpha \eta^\beta \\
& + [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w \cdot \nabla Z) + p|\nabla w|^{p-2} [(b'w_\alpha - c_\alpha b) \nabla w \cdot \nabla w_\beta \eta^\alpha \eta^\beta] \\
& + p|\nabla w|^{p-2} [(b'w_\beta - c_\beta b) \nabla w \cdot \nabla w_\alpha \eta^\alpha \eta^\beta] + [(c'w_\alpha - c_\alpha c) \nabla w \cdot \nabla w_\beta \eta^\alpha \eta^\beta] \\
& + [(c'w_\beta - c_\beta c) \nabla w \cdot \nabla w_\alpha \eta^\alpha \eta^\beta] - \epsilon \psi [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w \cdot \nabla |\eta|^2) \\
& [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w_\alpha \cdot \nabla w_\beta) \eta^\alpha \eta^\beta + p(p-2)b(w)|\nabla w|^{p-4} w_k w_l w_{k\alpha} w_{l\beta} \eta^\alpha \eta^\beta \\
& + [b'|\nabla w|^{p-2} + c'|\nabla w|^2 + d'] w_\alpha w_\beta \eta^\alpha \eta^\beta + [b'|\nabla w|^p + c'|\nabla w|^2 + d'] w_{\alpha\beta} \eta^\alpha \eta^\beta.
\end{aligned}$$

We are now going to choose  $c_\alpha$  such that

$$(2.6) \quad \rho' w_\alpha - 2c_\alpha \rho = 0$$

at  $(x_0, t_0)$ . Then, by the **conditions I** and (2.5),

$$\begin{aligned}
(2.7) \quad Z_{,t} - \epsilon \psi' |\eta|^2 \geq & \rho a^{ij} D_{ij} Z + \rho a_k^{ij} [w_{\alpha k} w_{\beta ij} \eta^\alpha \eta^\beta + w_{\beta k} w_{\alpha ij} \eta^\alpha \eta^\beta] \\
& - 2\rho a^{ij} c_j c_\alpha w_{, \beta i} \eta^\alpha \eta^\beta - 2\rho a^{ij} c_i c_\alpha w_{, \beta j} \eta^\alpha \eta^\beta \\
& - \epsilon \psi a^{ij} (|\eta|^2)_{ij} + \rho a_k^{ij} w_{ij} [Z_k - 2c_\alpha w_{\beta k} \eta^\alpha \eta^\beta - \epsilon \psi (|\eta|^2)_k] + \rho a_{kl}^{ij} w_{k\alpha} w_{l\beta} w_{ij} \eta^\alpha \eta^\beta \\
& + \rho' w_{\alpha\beta} a_{ij} w_{ij} \eta^\alpha \eta^\beta + \rho' a_k^{ij} w_\alpha w_{k\beta} w_{ij} \eta^\alpha \eta^\beta + \rho' a_{ij}^k w_\beta w_{k\alpha} w_{ij} \eta^\alpha \eta^\beta \\
& + [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w \cdot \nabla Z) + p|\nabla w|^{p-2} [(b'w_\alpha - c_\alpha b) \nabla w \cdot \nabla w_\beta \eta^\alpha \eta^\beta] \\
& + p|\nabla w|^{p-2} [(b'w_\beta - c_\beta b) \nabla w \cdot \nabla w_\alpha \eta^\alpha \eta^\beta] + [(c'w_\alpha - c_\alpha c) \nabla w \cdot \nabla w_\beta \eta^\alpha \eta^\beta] \\
& + [(c'w_\beta - c_\beta c) \nabla w \cdot \nabla w_\alpha \eta^\alpha \eta^\beta] - \epsilon \psi [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w \cdot \nabla |\eta|^2) \\
& [pb(w)|\nabla w|^{p-2} + 2c(w)] (\nabla w_\alpha \cdot \nabla w_\beta) \eta^\alpha \eta^\beta + p(p-2)b(w)|\nabla w|^{p-4} w_k w_l w_{k\alpha} w_{l\beta} \eta^\alpha \eta^\beta \\
& + [b'|\nabla w|^p + c'|\nabla w|^2 + d'] w_{\alpha\beta} \eta^\alpha \eta^\beta.
\end{aligned}$$

Suppose that the minimum of the second derivatives of  $w$  is taken along a direction  $\bar{\alpha}$  at  $(0, t_0)$ . Then,  $e_{\bar{\alpha}}$  is an eigen direction of the symmetric matrix  $D^2 w(0, t_0)$ . Thus, at  $(0, t_0)$ , we have

$$\begin{aligned}
Z &= w_{\bar{\alpha}\bar{\alpha}} + \epsilon \psi(t_0) = 0, \\
w_{\bar{\alpha}\beta} &= 0 \quad \text{if } \bar{\alpha} \neq \beta \text{ and } e_\beta \in \mathbb{R}^n, \\
D^2 Z &\geq 0, \quad \Delta Z \geq 0, \quad \nabla Z = 0 \quad \text{and} \quad Z_t \leq 0.
\end{aligned}$$

In addition, we get

$$(|\eta|^2)_{ij} = 2nc_{\bar{\alpha}}^2 + \frac{c_{\bar{\alpha}}(c_i + c_j)}{2} \quad \text{and} \quad \nabla_x w \cdot \nabla_x |\eta|^2 = 2c_{\bar{\alpha}} w_{\bar{\alpha}} \quad \text{at } (x_0, t_0).$$

Since  $a_{ij}$  is uniformly positive, by (2.7), we have

$$\begin{aligned}
 -\epsilon\psi' &\geq 2\rho a_{\alpha}^{ij}w_{\alpha ij}w_{\alpha\alpha} - 4\rho c_{\alpha}^2 a^{\alpha\alpha}w_{\alpha\alpha} - \epsilon\psi a^{ij}(|\eta|^2)_{ij} - 2c_{\alpha}\rho a_{\alpha}^{ij}w_{ij}w_{\alpha\alpha} \\
 &\quad - \rho a_k^{ij}w_{ij}\epsilon\psi(|\eta|^2)_k + \rho a_{\alpha\alpha}^{ij}w_{ij}w_{\alpha\alpha}^2 + \rho' a^{ij}w_{ij}w_{\alpha\alpha} + 2\rho' w_{\alpha} a_{\alpha}^{ij}w_{ij}w_{\alpha\alpha} \\
 &\quad + 2p|\nabla w|^{p-2}(b'w_{\alpha} - c_{\alpha}b)w_{\alpha}w_{\alpha\alpha} + 2(c'w_{\alpha} - c_{\alpha}c)w_{\alpha}w_{\alpha\alpha} \\
 &\quad - \epsilon\psi[pb|\nabla w|^{p-2} + 2c](\nabla w \cdot \nabla|\eta|^2) + [pb|\nabla w|^{p-2} + 2c]w_{\alpha\alpha}^2 + p(p-2)b|\nabla w|^{p-4}w_{\alpha}^2w_{\alpha\alpha}^2 \\
 &\quad + [b'|\nabla w|^p + c'|\nabla w|^2 + d']w_{\alpha\alpha} \\
 &:= -K\epsilon\psi - L\epsilon^2\psi^2
 \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
 K_1 &= 2\rho a_{\alpha}^{ij}w_{\alpha ij} - 4\rho c_{\alpha}^2 a^{\alpha\alpha} - 2c_{\alpha}\rho a_{\alpha}^{ij}w_{ij} + \rho' a^{ij}w_{ij} + 2\rho' w_{\alpha} a_{\alpha}^{ij}w_{ij} \\
 &\quad + 2p|\nabla w|^{p-2}(b'w_{\alpha} - c_{\alpha}b)w_{\alpha} + 2(c'w_{\alpha} - c_{\alpha}c)w_{\alpha} + b'|\nabla w|^p + c'|\nabla w|^2 + d' \\
 &\quad + a^{ij}\left(2nc_{\alpha}^2 + \frac{c_{\alpha}(c_i + c_j)}{2}\right) + 2c_{\alpha}w_{\alpha}[pb|\nabla w|^{p-2} + 2c]
 \end{aligned}$$

and

$$L = -\rho a_{\alpha\alpha}^{ij}w_{ij} - (pb|\nabla w|^{p-2} + 2c) - p(p-2)b|\nabla w|^{p-4}w_{\alpha}^2.$$

By the boundary condition, the minimum point  $0, t_0$  appears far away from the boundary. Hence, we can choose a compact subset  $\mathbb{S}$  in  $\Omega$  such that

$$(x_0(t), z_0(t)) \in \mathbb{S} \quad \text{and} \quad w(x, t) : \text{bounded above and below in } \mathbb{S}.$$

The equation (2.1), when restricted on  $\mathbb{S}$ , is nondegenerate. Therefore, the classical estimates for linear parabolic equations give

$$|Dw(0, t)| \leq C\|w\|_{L^\infty(\mathbb{S})} \quad \text{and} \quad |D^2w(0, t)| \leq C\|w\|_{L^\infty(\mathbb{S})}$$

for constant  $C > 0$ . Hence, the quantity  $K$  and  $L$  are under control, i.e., there exists a constant  $M$  such that

$$|K|, |L| \leq M \quad \text{at } (0, t_0).$$

Hence, if we choose the function  $\psi(t)$  having the property

$$\psi_t(t) > |K|\psi + |L|\psi^2,$$

then, contradiction arises. Therefore, the time  $t = t_0$  satisfying (2.3) doesn't exist. Thus,

$$\inf_{x \in \Omega, s \in [0, t]} \inf_{e_\beta \in \mathbb{R}^n, |e_\beta|=1} w_{\beta\beta} > -\epsilon \sup_{0 \leq s \leq t} \psi(s), \quad \forall t > 0.$$

Letting  $\epsilon \rightarrow 0$ , we can get a desired conclusion.  $\square$

### 3. GEOMETRIC PROPERTY OF GELFAND PROBLEM

In previous section, we discussed the positivity of the second derivatives of solutions for de-generated parabolic equation. It is a very useful tool for investigating the geometric properties of solutions to Gelfand's problem. We now address the long-time geometrical properties of solutions for the initial value problem with exponential growth:

$$(e^u)_t = \Delta u + \lambda e^u \quad \text{in } Q = \Omega \times (0, \infty) \tag{3.1}$$

posed in a strictly convex bounded domain  $\Omega \subset \mathbb{R}^n$  with

$$u = 0 \quad \text{on } \Omega, \quad u > 0 \text{ in } \Omega \tag{3.2}$$

and initial data

$$(3.3) \quad u(x, 0) = u_0(x) \leq \varphi(x)$$

where  $\varphi$  is the minimal solution of the Gelfand's problem  $(GP_\lambda)$ .

In this section, we aim at providing the  $f$ -convexity of the minimal solution  $\varphi$  to the problem  $(GP_\lambda)$  which also satisfies the boundary condition (1.4), i.e., we want to show that

$$e^{-\frac{1}{2}\varphi(x)} \quad : \quad \text{convex with respect to space variables.}$$

If we try to show the  $f$ -convexity of  $\varphi$  in  $(GP_\lambda)$ , we can put  $v = e^{-\frac{1}{2}\varphi}$  and replace  $\varphi$  by  $-2 \log v$  in the equation. Then

$$v \Delta v - |\nabla v|^2 - \frac{\lambda}{2} = 0.$$

By direct computation, we get

$$v \Delta w_{\alpha\alpha} + 2v_\alpha \Delta v_\alpha - 2\nabla v \cdot \nabla v_{\alpha\alpha} - 2\nabla v_\alpha \nabla v_\alpha = 0.$$

Unfortunately, there are many terms, for example  $\Delta v_\alpha$ , which are out of control. We don't even have any information about  $v_{\alpha\alpha}$ . Hence, inferring the geometric properties of  $\varphi$  from the equation  $(GP_\lambda)$  directly is very hard. However, according to the arguments in Lee and Vázquez's paper, [LV], the geometric properties of solutions to the nonlinear elliptic problem can be obtained by the geometric properties of the solution to the corresponding problem with parabolic flow. To apply their arguments to the Gelfand's problem, we start by showing the relation between the solution  $u$  of (3.1)-(3.3) and  $\varphi$  of  $(GP_\lambda)$ .

**Lemma 3.1 (Approximation lemma).** *Let  $u(x, t)$  be a solution of (3.1) and let  $\varphi$  be the minimal solution of  $(GP_\lambda)$ . Then, we have the following properties: For any sequence  $\{t_n\}_{n=1}^\infty$  with  $t_n \rightarrow \infty$ , we have a subsequence  $\{t_{n_k}\}_{k=1}^\infty$  such that*

$$(3.4) \quad \lim_{k \rightarrow \infty} |u(x, t_{n_k}) - \varphi(x)| \rightarrow 0$$

uniformly in compact subset of  $\Omega$ .

*Proof.* Define the functional  $F(\phi)$  by

$$F(\phi) = - \int_{\Omega} \left( \frac{1}{2} \phi \Delta \phi + \lambda e^\phi \right) dx$$

and let  $g(t) = F(u(\cdot, t))$ . Then a simple computation yields

$$(3.5) \quad g'(t) = -\lambda \int_{\Omega} e^{u(x,t)} [u_t(x, t)]^2 dx \leq 0.$$

Uniformly ellipticity of the coefficients in (3.1) show that  $\int_{\Omega} e^{u(x,t)} dx$  is bounded for all  $t \geq 0$ . Hence, for some constant  $M > 0$ ,

$$(3.6) \quad g(t) > -M \quad \forall t > 0.$$

Therefore, by (3.5) and (3.6),  $\lim_{t \rightarrow \infty} g(t)$  exists and  $g'(t) \rightarrow 0$ . Hence, for any sequence of times  $\{t_n\}_{n=1}^\infty$ ,  $t_n \rightarrow \infty$ ,  $g'(t_n) \rightarrow 0$ .

Observe that the equation (3.1) is uniformly parabolic in  $\Omega$ . Thus, by the comparison principle, there exists a uniform constant  $C$  such that

$$(3.7) \quad 0 \leq u(x, t) \leq \varphi(x) \leq C.$$

Moreover, by the Schauder estimates for parabolic equation [LSU], the sequence  $u(\cdot, t_n)$  is equi-Hölder continuous on every compact subset  $K$  of  $\Omega$ . Hence, by the Ascoli Theorem, the sequence



$\{u(\cdot, t_n)\}$  has a subsequence  $\{u(\cdot, t_{n_k})\}$  that converges to some function  $h$  uniformly on every compact subset of  $K$  which is non-trivial. The readers can easily check the non-triviality of  $h$ .

Multiplying equation (3.1) by any test function  $\eta \in C_0^\infty(\Omega)$  and integrate in space. Then for each  $t_{n_k}$ ,

$$(3.8) \quad \lambda \int_K e^u u_t \eta dx = \int_K u \Delta \eta dx + \lambda \int_K e^u \eta dx.$$

Since the absolute value of the left hand side of (3.8) is bounded above by

$$(3.9) \quad \lambda \left( \int_\Omega \eta^2 e^u dx \right)^{\frac{1}{2}} \left( \int_\Omega e^u (u_t)^2 dx \right)^{\frac{1}{2}}$$

and the second term of (3.9) has limit zero as  $t_{n_k} \rightarrow \infty$ , we get in the limit  $t_{n_k} \rightarrow \infty$

$$0 = \int_K h \Delta \eta dx + \lambda \int_K e^h \eta dx,$$

which is weak formulation of the equation

$$(3.10) \quad \Delta h + \lambda e^h = 0 \quad \text{in } K.$$

By the arbitrary choice of a compact subset  $K$  in  $\Omega$ , (3.10) holds in  $\Omega$ . Therefore,  $h$  is a weak solution of the Gelfand's problem which is smaller than the minimal solution  $\varphi$  because of (3.7). Since  $\varphi$  is the minimal solution of Gelfand's problem,  $u(x, t_{n_k})$  converges uniformly on every compact subset of  $\Omega$  to  $\varphi$  as  $t_{n_k} \rightarrow \infty$  and the lemma follows.  $\square$

**3.1. geometric property.** We first establish some estimates for the solution  $u$  of (3.1) which plays an important role for the geometric properties of solution  $u$  on the boundary.

**Lemma 3.2.** *Let  $u \in C^2(\Omega) \times (0, \infty)$  be a solution of (3.1)-(3.3). Then, there exists a constant  $C > 0$  such that*

$$|u_{\tau_k v}(x, t)| \leq C |u_v(x, t)|, \quad \forall x \in \partial\Omega \quad (k = 1, \dots, n-1)$$

where  $v$  and  $\tau_k$  are normal and tangential directions to  $\partial\Omega$  at  $(x, t)$ , respectively.

*Proof.* Without loss of generality, we may assume that  $x = 0$  and outer normal direction  $e_v = e_n$ . For any  $1 \leq k \leq n-1$ , we consider the directional derivative

$$\partial_{T_k} u = P(x) \cdot \nabla u, \quad (P(x) = (p_1(x), \dots, p_n(x)))$$

which is the same as a tangential derivative on the boundary  $\partial\Omega$ . By the strictly convexity of domain  $\Omega$ , we can find the largest circle of radius  $R = R_k$  that touches the domain  $\Omega$  from inside at  $x = 0$  in the plane generated by two directions  $e_{\tau_k}$  and  $e_v$ . Note that the reciprocal of  $R$  lies on between principle curvatures at  $x = 0$ . Moreover,  $(R - x_n)u_k + x_k u_n$  is a tangential derivative on the circle. Thus, the directional derivative  $\partial_{T_k} u$  can be expressed in the form

$$\partial_{T_k} u(x, t) = (R - x_n + O(|x|^2)) u_k(x, t) + (x_k + O(|x|^2)) u_n(x, t)$$

near the point  $x = 0$ .

For a constant  $0 < \rho < 1$  and a time  $t_0 > 0$ , let  $B_\rho = B_\rho(0)$  be the ball of radius  $\rho$  centered at 0 and define the new functions  $h_\pm$  and  $H_\pm$  by

$$(3.11) \quad \begin{aligned} h_\pm &= \partial_{T_k} u \pm \sum_{l=1}^{n-1} (\partial_{T_l} u)^2, \\ H_\pm &= \eta^2 h_\pm \pm \mu x_n^2, \end{aligned}$$



where  $\eta(x, t) = (\rho^2 - |x|^2)(t - t_0 + \rho^2)$ . Then, it can be easily checked that

$$(3.12) \quad |H_{\pm}(x, t)| \leq -\frac{\mu\rho^2}{2}x_n.$$

on  $\{\partial(\Omega \cap B_\rho)\} \times [t_0 - \rho^2, t_0]$  and  $\{\Omega \cap B_\rho\} \times \{t = t_0 - \rho^2\}$

Let us now show that  $LH_+ \geq 0$  and  $LH_- \leq 0$  in  $\Omega \cap B_\rho$  where  $L$  is defined by

$$Lf = \Delta f + \lambda e^u f - f_t.$$

By direct computation, we can find a constant  $\rho_1 > 0$  such that

$$\sum_{l=1}^{n-1} |\partial_{T_l} u|^2 \geq \frac{R^2}{4} \sum_{l=1}^{n-1} |\nabla u_l|^2 - C_1 (1 + |\nabla u|^2) \quad \forall 0 < \rho \leq \rho_1$$

for some constant  $C_1 > 0$ . Thus, we have, for some constant  $c > 0$ ,

$$\begin{aligned} LH_+ &\geq \eta^2 (\Delta h_+ + \lambda e^u h_+ - (h_+)_t) + 2h_+ (\eta \Delta \eta + |\nabla \eta|^2 - \eta \eta_t) \\ &\quad + 4\eta \nabla \eta \cdot \nabla h_+ + 2\mu \\ &\geq 2\eta^2 \sum_{l=1}^{n-1} |\nabla \partial_{T_l} u|^2 \\ &\quad - c\eta^2 \left[ 1 + 2(R|\nabla_{x'} u| + \rho|u_n|) \right] \left[ \sum_{l=1}^{n-1} \left\{ |\nabla u| + \rho \left( \sum_{i=1}^n |u_{li}| + \sum_{i=1}^{n-1} |u_{in}| + |\Delta_{x'} u| + \lambda e^u + |u_t| \right) \right\} \right] \\ &\quad - \lambda \eta^2 e^u (R^2 |\nabla_{x'} u|^2 + 2R\rho |\nabla_{x'} u| |u_n| + \rho^2 |u_n|^2) \\ &\quad - 4n(n+3)\rho^6 (R|\nabla_{x'} u| + \rho|u_n| + R^2 |\nabla_{x'} u|^2 + 2R\rho |\nabla_{x'} u| |u_n| + \rho^2 |u_n|^2) \\ &\quad - 16\rho^3 \eta (R|\nabla_{x'} u| + \rho|u_n|) \sum_{l=1}^{n-1} |\nabla (\partial_{T_l} u)| \\ &\quad - 8\rho^4 \eta \left[ \sum_{i=1}^{n-1} |u_{ni}| + |\Delta_{x'} u| + \lambda e^u + |u_t| \right] - 8R\rho^3 \eta \sum_{i=1}^{n-1} |u_{ki}| - 8\rho^3 \eta |\nabla u| + 2\mu \\ &\geq \left[ \eta^2 \sum_{l=1}^{n-1} |\nabla \partial_{T_l} u|^2 - 16\rho^3 \eta (R|\nabla_{x'} u| + \rho|u_n|) \sum_{l=1}^{n-1} |\nabla (\partial_{T_l} u)| + \frac{\mu}{2} \right] \\ &\quad + \left[ \frac{R^2 \eta^2}{8} \sum_{l=1}^{n-1} |\nabla u_l|^2 - 3c\rho \eta^2 \left[ 1 + 2(R|\nabla_{x'} u| + \rho|u_n|) \right] \sum_{l=1}^{n-1} |\nabla u_l| + \frac{\mu}{2} \right] \\ &\quad + \left[ \frac{R^2 \eta^2}{8} \sum_{l=1}^{n-1} |\nabla u_l|^2 - 8\rho^3 \eta (R + \rho) \sum_{l=1}^{n-1} |\nabla u_l| + \frac{\mu}{2} \right] \\ &\quad + \frac{\mu}{2} - 2\rho^6 [K_0 + K_1 \rho + K_2 \rho^2 + K_3 \rho^3 + K_4 \rho^4] \end{aligned}$$

where

$$K_0 = 4n(n+3) (R|\nabla_{x'} u| + R^2 |\nabla_{x'} u|^2)$$

$$K_1 = 4 [n(n+3) (|u_n| + 2R|\nabla_{x'} u| |u_n|) + 2|\nabla u|],$$

$$K_2 = \frac{C_1}{2} (1 + |\nabla u|^2) + 4n(n+3)|u_n|^2 + \lambda R^2 e^u |\nabla_{x'} u| + 8(\lambda e^u + |u_t|) + c(n-1)(1 + 2R|\nabla_{x'} u|)|\nabla u|,$$

$$K_3 = c(n-1) [|u_n| |\nabla u| + (1 + 2R |\nabla_{x'} u|) (\lambda e^u + |u_t|)] + 2R \lambda \rho e^u |\nabla_{x'} u| |u_n|$$

and

$$K_4 = c(n-1) |u_n| (\lambda e^u + |u_t|) + \lambda e^u |u_n|.$$

We now choose a constant  $\rho_2$  such that,  $\forall 0 < \rho \leq \rho_2$ ,

$$\rho |\nabla u|^2 \leq 1, \quad 2^7 (R |\nabla_{x'} u| + \rho |u_n|)^2 < |u_n(0, t_0)|,$$

$$\frac{36c^2 \rho^4}{R^2} \left[ 1 + 2 (R |\nabla_{x'} u| + \rho |u_n|) \right]^2 \leq |u_n(0, t_0)|, \quad 2^8 \left( 1 + \frac{\rho}{R} \right)^2 \leq 2^9$$

and

$$K_0 + K_1 \rho + K_2 \rho^2 + K_3 \rho^3 + K_4 \rho^4 < \frac{|u_n(0, t_0)|}{4}$$

in  $\{\Omega \cap B_\rho\} \times [t_0 - \rho^2, t_0]$ . Thus, for  $0 < \rho < \min\{\rho_1, \rho_2\}$ ,

$$(3.13) \quad L(H_+) \geq 0 \quad \text{if } \mu \geq \max\{|u_n(0, t_0)| \rho^6, 2^9 \rho^6\} = 2C_1 |u_n(0, t_0)| \rho^6.$$

Hence, by (3.12) and (3.13),

$$H_+(x, t) \leq -C_1 |u_n(0, t_0)| \rho^8 x_n \quad \text{on } \Omega \cap B_\rho \times (t_0 - \rho^2, t_0].$$

This immediately implies that

$$h_+(x, t) \leq C_1 u_n(0, t_0) x_n \quad \text{in } \Omega \cap B_{\frac{\rho}{2}} \times \left(t_0 - \frac{\rho^2}{4}, t_0\right]$$

Similarly, we can also show that

$$h_-(x, t) \geq -C_2 u_n(0, t_0) x_n \quad \text{in } \Omega \cap B_{\frac{\rho}{2}} \times \left(t_0 - \frac{\rho^2}{4}, t_0\right]$$

for some constant  $C_2 > 0$ . Therefore

$$-\bar{C} R u_n(0, t_0) x_n \leq v_-(x, t) \leq v_+(x, t) \leq \bar{C} R u_n(0, t_0) x_n \quad \text{in } \Omega \cap B_{\frac{\rho}{2}} \times \left(t_0 - \frac{\rho^2}{4}, t_0\right].$$

for constant  $\bar{C} = \frac{1}{R} \max\{C_1, C_2\}$ . Taking the normal derivative  $\partial_n$  at  $(0, t_0)$ , we obtain

$$|u_{kn}(0, t_0)| = \left| \frac{(v_+)_n(0, t_0)}{R} \right| \leq \bar{C} |u_n(0)|, \quad \forall 1 \leq k \leq n-1$$

and the lemma follows.  $\square$

Next, coming to our subject, we have the following result about *preservation of  $f$ -convexity*, which is easy but allows to present the basic technique. Our geometrical results will be derived under the extra assumption that  $\Omega$  is strictly convex.

**Lemma 3.3.** *Let  $\Omega$  be a strictly convex bounded subset in  $\mathbb{R}^n$  and assume that  $u$  is a solution of (3.1)-(3.3) with the boundary condition (1.4). Let  $u = -2 \log w$ . Then, for every  $t > 0$ , as  $x \rightarrow x_0 \in \partial\Omega$*

$$(3.14) \quad w_{\alpha\alpha}(x, t) = \frac{1}{2} e^{-\frac{1}{2}u} \left( \frac{1}{2} u_\alpha^2 - u_{\alpha\alpha} \right) \geq \delta_0 > 0, \quad (\alpha = e_\alpha \in \mathbb{R}^n, \quad \text{and} \quad |e_\alpha| = 1)$$

for a uniform constant  $\delta_0$  depending on  $\partial\Omega$ .

*Proof.* By direct computation, we have

$$w_{\alpha\alpha}(x, t) = \frac{1}{2}e^{-\frac{1}{2}u} \left( \frac{1}{2}u_\alpha^2 - u_{\alpha\alpha} \right), \quad \text{on } \partial\Omega.$$

If  $e_\alpha = \tau$ , a tangent direction at  $x_0$  to  $\partial\Omega$ , then  $u_\alpha = 0$ . Hence, we need to estimate  $u_{\alpha\alpha}$ . For this, we use the fact that  $\partial\Omega$  is strictly convex. Without loss of generality, we may assume that  $x_0 = 0$  and the tangent plane is  $x_n = 0$ . We may also assume that the boundary is given locally by the equation  $x_n = f(x')$ , and  $x' = (x_1, \dots, x_{n-1})$ . We introduce the change of variables

$$(3.15) \quad \tilde{x}_\alpha = x_\alpha, \quad \tilde{x}_n = x_n - f(x'), \quad g(\tilde{x}', \tilde{x}_n, t) = u(x', x_n, t).$$

Then along tangent directions we have

$$\begin{aligned} u_{\alpha\alpha}(x', x_n, t) &= g_{\alpha\alpha}(\tilde{x}', \tilde{x}_n, t) - 2g_{\alpha\tilde{x}_n}(\tilde{x}', \tilde{x}_n, t)f_\alpha(x') \\ &\quad + g_{\tilde{x}_n\tilde{x}_n}(\tilde{x}', \tilde{x}_n, t)(f_\alpha(x'))^2 - g_{\tilde{x}_n}(\tilde{x}', \tilde{x}_n, t)f_{\alpha\alpha}(x'). \end{aligned}$$

Since  $f_\alpha(0) = 0$  and  $f_{\alpha\alpha}(0)$  are nonzeros along all tangent directions, we get

$$(3.16) \quad u_{\alpha\alpha}(0, 0, t) = -g_{\tilde{x}_n}(0, 0, t)f_{\alpha\alpha}(0) = -u_{x_n}(0, 0, t)f_{\alpha\alpha}(0).$$

Since  $u$  is a solution to a parabolic equation with uniformly elliptic coefficients, by the Hopf's lemma for the classical partial differential equation,  $0 < c_0 \leq |\nabla u|$ . Hence

$$u_{\tau\tau}(x_0, t) = -u_\nu(x_0, t)f_{\tau\tau}(0) < 0$$

and

$$(3.17) \quad w_{\tau\tau}(x, t) = \frac{1}{2}e^{-\frac{1}{2}u} \left( \frac{1}{2}u_\tau^2 - u_{\tau\tau} \right) > \frac{c_0 f_{\tau\tau}(0)}{2} \quad \text{as } x \rightarrow x_0 \in \partial\Omega.$$

Let  $e_\alpha = \nu$ . On  $\partial\Omega$

$$0 = \lambda (e^u)_t = \Delta u + \lambda e^u = \Delta u + \lambda.$$

Thus, we have

$$\frac{1}{2}u_\nu^2 - u_{\nu\nu} = \frac{1}{2}u_\nu^2 + \lambda + \sum_{i=1}^{n-1} u_{\tau_i\tau_i} = \frac{1}{2}u_\nu^2 + \lambda + (n-1)u_\nu H(\partial\Omega) \quad \text{on } \partial\Omega,$$

where  $H(\partial\Omega)$  is the mean curvature of  $\partial\Omega$  at  $x_0$ . By the boundary condition (1.4), we can obtain

$$(3.18) \quad w_{\nu\nu}(x, t) = \frac{1}{2}e^{-\frac{1}{2}u} \left( \frac{1}{2}u_\nu^2 - u_{\nu\nu} \right) = \frac{1}{2} \left( \frac{1}{2}u_\nu^2 + \lambda + (n-1)u_\nu H(\partial\Omega) \right) > 0 \quad \text{on } \partial\Omega.$$

Finally, we check the case that the minimum of the second derivatives of  $w$  occurs along a general direction  $e_\alpha$  at  $x_0$ . Since the outer normal direction  $e_\nu$  is vertical to the tangent plane, we can express a general direction  $e_\alpha$  by

$$(3.19) \quad e_\alpha = k_1 e_{\alpha_\tau} + k_2 e_\nu, \quad (k_1^2 + k_2^2 = 1).$$

where  $e_{\alpha_\tau}$  is a direction contained on the tangent plane to the graph of  $u$  at  $(0, t)$ . Hence, the second derivatives of  $w$  can be written in the form

$$\begin{aligned} w_{\alpha\alpha} &= k_2^2 w_{\nu\nu} + 2k_1 k_2 w_{\alpha_\tau\nu} + k_1^2 w_{\alpha_\tau\alpha_\tau} \\ &= \frac{1}{2}e^{\frac{1}{2}u} \left[ k_2^2 \left( \frac{1}{2}u_\nu^2 - u_{\nu\nu} \right) + 2k_1 k_2 \left( \frac{1}{2}u_{\alpha_\tau\nu} - u_{\alpha_\tau\nu} \right) + k_1^2 \left( \frac{1}{2}u_{\alpha_\tau}^2 - u_{\alpha_\tau\alpha_\tau} \right) \right]. \end{aligned}$$

Thus, at  $x = x_0$ , we have

$$(3.20) \quad w_{\alpha\alpha} = \frac{1}{2}e^{\frac{1}{2}u} \left[ \frac{k_2^2}{2}u_\nu^2 - k_2^2 u_{\nu\nu} - 2k_1 k_2 u_{\alpha_\tau\nu} - k_1^2 u_{\alpha_\tau\alpha_\tau} \right].$$

By Lemma 3.2, there exists some constant  $C_1 > 0$  such that

$$(3.21) \quad |u_{\alpha_\tau v}(0, t)| = |v_{\alpha_\tau v}(0, t)| \leq C_1 |v_v(0, t)| = C_2 |u_v(0, t)|$$

Combining (3.21) with (3.20), we can get

$$w_{\alpha\alpha} = \frac{e^{-\frac{1}{2}u}}{2} \left[ k_2^2 \left( \frac{1}{2} u_v^2 + \lambda + (n-1)H(\partial\Omega)u_v \right) - C_1 k_1 k_2 |u_v| + k_1^2 f_{\alpha_\tau \alpha_\tau} u_v \right] \quad \text{at } x = x_0.$$

Therefore, if  $u$  satisfies the boundary condition (1.4) for

$$K \geq \frac{C_1^2}{4 |f_{\alpha_\tau \alpha_\tau}(0)|},$$

then the result (3.14) holds for all  $e_\alpha \in \mathbb{R}^n$ , ( $|e_\alpha| = 1$ ) and the lemma follows.  $\square$

**Theorem 3.4.** *Let  $\Omega$  be a convex bounded domain and let  $u_0 \geq 0$  be a continuous and bounded initial function which satisfies (3.3). Then, the solution  $u$  of (3.1)-(3.3) with the boundary condition (1.4) is  $f$ -convex in the space variable for all  $t \geq 0$ , i.e.,*

$$(3.22) \quad D^2 \left( e^{-\frac{1}{2}u} \right) \geq 0.$$

*Proof.* Let  $w = e^{-\frac{1}{2}u}$ . Then, the new function  $w$  satisfies

$$(3.23) \quad \lambda w_t = w^2 \Delta w - w |\nabla w|^2 - \frac{\lambda}{2} w.$$

Hence,  $w$  is a solution to the equation (2.1) with  $\rho(w)$ ,  $a_{ij}(\nabla w)$ ,  $b(w)$ ,  $c(w)$  and  $d(w)$  being replaced by  $w^2$ ,  $I_n$ ,  $0$ ,  $-w$  and  $-\frac{\lambda}{2}w$  respectively. Here,  $I_n$  is the  $n \times n$  identity matrix.

We now are going to check that the condition **I.1** and **I.2** given in Section 2 hold for the coefficients in the equation (3.23). It is trivial that

$$b(w) = 0, \quad c(w) = -w, \quad d(w) = -\frac{\lambda}{2}w : \quad \text{convex.}$$

In addition, by direct computation, it can be easily shown that

$$\rho'' - \frac{(\rho')^2}{2\rho} = 2 - \frac{4w^2}{2w^2} = 2 - 2 = 0.$$

Hence, the equation (3.23) have the condition **I** and **II**. On the other hand, Lemma 3.3 tells us that the solution of the equation (3.23) with zero boundary condition is convex on the boundary. Therefore, by Lemma 2.1, it is also convex in the interior of domain  $\Omega$  and the lemma follows.  $\square$

**Corollary 3.5.** *If  $\Omega$  is convex, the stationary profile  $\varphi(x)$  of  $u(x, t)$  is  $f$ -convex, i.e.,  $D^2 \left[ e^{-\frac{1}{2}\varphi(x)} \right] \geq 0$ .*

*Proof.* Take the initial data as before. By the asymptotic result, Lemma 3.1, we have uniform convergence between  $\varphi(x)$  and  $u(x, t)$ . Hence, the conclusion follows.  $\square$

Note that  $\varphi$  satisfies the equation

$$\Delta \varphi + \lambda e^\varphi = 0 \quad \text{in } \Omega.$$

Then  $\bar{w} = f(\varphi)^2 = e^{-\varphi}$  satisfies

$$\bar{w} \Delta \bar{w} - |\nabla \bar{w}|^2 - \lambda \bar{w} = 0 \quad \text{in } \Omega.$$

It has the similar form to the equation in (3.12) of the paper [LV]. Hence, following the same arguments as in the proof of Lemma 3.6 in [LV], we get an improved result.

**Lemma 3.6** (Strictly  $f$ -convexity). *If  $\Omega$  is smooth and strictly convex, then the minimal solution  $\varphi(x)$  is strictly  $f$ -convex: there exists a constant  $c_1 > 0$  such that*

$$D^2 f(\varphi) = D^2 \left( e^{-\frac{1}{2}\varphi} \right) \leq -c_1 I.$$

*The constant  $c_1$  depends only on the shape of  $\Omega$ .*

#### 4. BOUNDARY CONDITION FOR THE GEOMETRIC PROPERTIES

Through the previous sections, we investigated the  $f$ -convexity of the minimal solution of Gelfand's problem,  $(GP_\lambda)$ , under the assumption that the minimal solution has the property (1.4) which we call **boundary condition**. As mentioned before, if we have the boundary condition removed from our setting, it is very difficult to get the  $f$ -convexity of the minimal solution of  $(GP_\lambda)$  on the boundary. Thus, we had no choice but to add the boundary condition (1.4) for the result. However, the problem is that we couldn't guarantee the existence of the minimal solution of  $(GP_\lambda)$  having the boundary condition (1.4). Therefore, we need to check whether it is possible to satisfy conditions,  $(GP_\lambda)$  and (1.4), simultaneously. Otherwise, it is meaningless to apply our geometric results in physical models.

Before we finish this work, we will introduce some properties related the boundary condition (1.4) in this last section of the paper. We are now ready to state and prove the main result in this section.

**Lemma 4.1.** *Let  $\Omega$  be a ball with radius  $r > 0$ , i.e.,  $\Omega = B_r$ , and let  $\psi_\lambda$  and  $\phi_\lambda$  be solutions of the Gelfand's problem  $(GP_\lambda)$  in a ball  $B_r$ .*

(1) *Suppose that  $\psi_\lambda \geq \phi_\lambda$ . Then, there exists a constant  $r_0 > 0$  such that if  $G(\phi_\lambda, \lambda, B_r) > 0$  then*

$$G(\psi_\lambda, \lambda, B_r) > 0 \quad \forall r > r_0.$$

(2) *Suppose that  $\psi_\lambda \geq \phi_\lambda$ . Then, there exists a constant  $r_1 > 0$  such that if  $G(\psi_\lambda, \lambda, B_r) \leq 0$  then*

$$G(\phi_\lambda, \lambda, B_r) \leq 0 \quad \forall r > r_1.$$

(3) *For any  $0 < \lambda < \lambda^*$ , there exist a constant  $r_2(\lambda) = r_2 > 0$  such that*

$$(4.1) \quad G(\psi_\lambda, \lambda, B_r) > 0 \quad \forall r > r_2.$$

(4) *For any  $0 < \lambda < \lambda^*$ , there exist a constant  $r_3(\lambda) = r_3 > 0$  and a solution  $\psi_\lambda$  of the Gelfand's problem defined on  $B_{r_3}$  such that  $G(\psi_\lambda, \lambda, B_{r_3}) > 0$  fails.*

*Proof.* For the convenient, we assume in this proof that  $B_r = B_r(0)$  a ball of radius  $r$  which is centered at 0. Define the function  $\theta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\theta_\lambda(x) = \frac{\lambda}{2n} (r^2 - |x|^2)$ . Then, by a direct computation

$$\Delta \theta(x) = -\lambda.$$

For any solution  $\phi_\lambda$  of the Gelfand's problem  $(GP_\lambda)$ , by the maximum principle for the super-harmonic function, we have  $\phi_\lambda \geq \theta_\lambda$  in  $B_r$  and  $\phi_\lambda = \theta_\lambda$  on  $\partial B_r$ . This immediately implies that

$$(4.2) \quad \phi_{\lambda,v} \leq -\frac{\lambda r}{n} \quad \text{on } \partial B_r.$$

(1) By assumption, we have

$$\psi_\lambda \geq \phi_\lambda \geq \theta_\lambda \quad \text{in } B_r \quad \text{and} \quad \psi_\lambda = \phi_\lambda = \theta_\lambda \quad \text{on } \partial B_r.$$

Thus, it follows that

$$\psi_{\lambda,v} \leq \phi_{\lambda,v} \leq -\frac{\lambda r}{n} \quad \text{on } \partial B_r.$$

where  $\nu$  is the outer normal vector at the boundary  $\partial B_r$ . Moreover, it also follows from  $G(\phi_\lambda, \lambda, B_r) > 0$  that the mean curvature  $H(B_r)$  satisfies

$$(n-1)H(\partial B_r) + K < -\frac{1}{2}\phi_{\lambda,\nu} - \frac{\lambda}{\phi_{\lambda,\nu}} \quad \text{on } \partial B_r.$$

Thus, for  $r > \frac{2n^2}{\lambda}$ ,

$$\begin{aligned} G(\psi_\lambda, \lambda, B_r) &= \frac{1}{2}(\psi_{\lambda,\nu})^2 + \lambda + (n-1)\psi_{\lambda,\nu}H(\partial\Omega) + K\psi_{\lambda,\nu} \geq \frac{1}{2}\psi_{\lambda,\nu}(\psi_{\lambda,\nu} - \phi_{\lambda,\nu}) + \lambda\left(1 - \frac{\psi_{\lambda,\nu}}{\phi_{\lambda,\nu}}\right) \\ &= \left(\frac{\psi_{\lambda,\nu} - \phi_{\lambda,\nu}}{\phi_{\lambda,\nu}}\right)\left(\frac{1}{2}\psi_{\lambda,\nu}\phi_{\lambda,\nu} - \lambda\right) \geq \lambda\left(\frac{\psi_{\lambda,\nu} - \phi_{\lambda,\nu}}{\phi_{\lambda,\nu}}\right)\left(\frac{\lambda r^2}{2n^2} - 1\right) > 0. \end{aligned}$$

Hence,  $G(\psi_\lambda, \lambda, B_r) > 0$  holds for  $r_0 = \sqrt{\frac{2n^2}{\lambda}}$ .

(2) In this case, the mean curvature  $H(B_r)$  satisfies

$$(n-1)H(\partial B_r) + K \geq -\frac{1}{2}\psi_{\lambda,\nu} - \frac{\lambda}{\psi_{\lambda,\nu}} \quad \text{on } \partial B_r.$$

Thus,  $G(\phi_\lambda, \lambda, B_r) \leq 0$  also holds for  $r_1 = \sqrt{\frac{2n^2}{\lambda}}$  since

$$\begin{aligned} G(\phi_\lambda, \lambda, B_r) &= \frac{1}{2}(\phi_{\lambda,\nu})^2 + \lambda + (n-1)\phi_{\lambda,\nu}H(\partial\Omega) + K\phi_{\lambda,\nu} \leq \frac{1}{2}\phi_{\lambda,\nu}(\phi_{\lambda,\nu} - \psi_{\lambda,\nu}) + \lambda\left(1 - \frac{\phi_{\lambda,\nu}}{\psi_{\lambda,\nu}}\right) \\ &= \left(\frac{\phi_{\lambda,\nu} - \psi_{\lambda,\nu}}{\psi_{\lambda,\nu}}\right)\left(\frac{1}{2}\psi_{\lambda,\nu}\phi_{\lambda,\nu} - \lambda\right) \leq \lambda\left(\frac{\psi_{\lambda,\nu} - \phi_{\lambda,\nu}}{\phi_{\lambda,\nu}}\right)\left(\frac{\lambda r^2}{2n^2} - 1\right) \leq 0. \end{aligned}$$

(3) Suppose that  $\psi_\lambda$  is a solution of the Gelfand's problem. Then, it follows from (4.2) that

$$\psi_{\lambda,\nu} \leq -\frac{\lambda r}{n} \quad \text{on } \partial B_r.$$

Therefore

$$\begin{aligned} G(\psi_\lambda, \lambda, B_r) &= \frac{1}{2}\psi_{\lambda,\nu}[\psi_{\lambda,\nu} + 2(n-1)H(B_r) + K] + \lambda \geq \frac{1}{2}\psi_{\lambda,\nu}\left[-\frac{\lambda r}{n} + \frac{2(n-1)}{r} + K\right] + \lambda \\ &> 0 \quad \text{if } r \geq \sqrt{\frac{2n(n-1)}{\lambda}} + \frac{nK}{\lambda}. \end{aligned}$$

Hence, (4.1) holds for  $r_2 = \sqrt{\frac{2n(n-1)}{\lambda}} + \frac{nK}{\lambda}$ .

(4) By a simple computation, it is easily checked that the function  $g(x) = \log\left(\frac{2(n-2)}{\lambda|x|^2}\right)$  is a solution of Gelfand's problem on  $B_{\sqrt{\frac{2(n-2)}{\lambda}}}$ . In addition, we also have

$$\frac{1}{2}(g_\nu)^2 + \lambda + (n-1)g_\nu H(B_r) + Kg_\nu = \frac{2\lambda}{2(n-2)} + \lambda - \frac{2(n-1)\lambda}{2(n-2)} + Kg_\nu \leq 0 \quad \text{on } \partial B_{\sqrt{\frac{2(n-2)}{\lambda}}}.$$

Hence,

$$G\left(\log\left(\frac{2(n-2)}{\lambda|x|^2}\right), \lambda, B_{\sqrt{\frac{2(n-2)}{\lambda}}}\right) \leq 0.$$

This finishes the proof of (4) and the lemma follows.  $\square$

We give next the generalization of the previous lemma for a strictly convex bounded domain  $\Omega$  with smooth boundary. For  $x \in \partial\Omega$ , we denote by

$$r_\Omega(x) = \sup\{r \in \mathbb{R} : B_r \subset \Omega \text{ and } B_r \cap \partial\Omega = \{x\}\} \quad \text{and} \quad r_\Omega = \inf_{x \in \partial\Omega} r_\Omega(x).$$

Then, we easily conclude that

$$H(\partial\Omega) \leq \frac{1}{r_\Omega}$$

and

$$\psi_\lambda \leq -\frac{\lambda r_\Omega}{n} \quad \text{on } \partial\Omega$$

for any solution  $\psi_\lambda$  of the Gelfand's problem  $(GP_\lambda)$ . Hence, following the similar computation as in the proof of Lemma 4.1, we can easily extend the results for the boundary condition on a ball to more general boundary setting.

We finish this work with stating the following result.

**Theorem 4.2.** *Let  $\Omega$  be a strictly convex bounded subset of  $\mathbb{R}^n$  and let  $\psi_\lambda$  and  $\phi_\lambda$  be solutions of the Gelfand's problem  $(GP_\lambda)$  in  $\Omega$ .*

(1) *Suppose that  $\psi_\lambda \geq \phi_\lambda$ . Then, there exists a constant  $r_0 > 0$  such that if  $G(\phi_\lambda, \lambda, \Omega) > 0$  then*

$$G(\psi_\lambda, \lambda, \Omega) > 0 \quad \forall r_\Omega > r_0.$$

(2) *Suppose that  $\psi_\lambda \geq \phi_\lambda$ . Then, there exists a constant  $r_1 > 0$  such that if  $G(\psi_\lambda, \lambda, \Omega) \leq 0$  then*

$$G(\phi_\lambda, \lambda, \Omega) \leq 0 \quad \forall r_\Omega > r_1.$$

(3) *For any  $0 < \lambda < \lambda^*$ , there exist a constant  $r_2 > 0$  such that*

$$G(\psi_\lambda, \lambda, \Omega) > 0 \quad \forall r_\Omega > r_2.$$

**Acknowledgement** Ki-Ahm Lee was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD, Basic Research Promotion Fund)( KRF-2008-314-C00023).

## REFERENCES

- [BV] H. BREZIS, J.L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Compl. Madrid 10 (1997) 443-469
- [Ch] S. Chandrasekhar, *An introduction to the study of stellar structure*, Dover Publ. Inc. 1985
- [CLMP] E. Caglioti, P.L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations*
- [Ge] I.M. Gelfand, *Some problems in the theory of quasi-linear equations*, Section 15, due to G.I. Barenblatt, American Math. Soc. Transl. 29 (1963), 295-381; Russian original: Uspekhi Mat. Nauk 14 (1959), 87-158
- [KC] H.B. Keller, D.S. Cohen, *Some positive problems suggested by nonlinear heat generation* J. Math. Mech. 16 (1967), 1361-1376
- [LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Mono. vol. 23, Amer. Math. Soc., Providence, R.L., USA, 1968
- [LV] K.-A. Lee, J.L. Vázquez *Parabolic approach to nonlinear elliptic eigenvalue problems*, Adv. Math. 219 (2008), no. 6, 2006-2028.
- [Ne] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations* C.R. Acad. Sci. Paris, Sér. I Math. 330 (2000) 997-1002



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